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# Exponential moments for planar tessellations

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# Exponential moments for planar tessellations

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**Abstract**

In this paper we show existence of all exponential moments for the total edge length in a unit disc for a family of planar tessellations based on Poisson point processes. Apart from classical such tessellations like the Poisson–Voronoi, Poisson–Delaunay and Poisson line tessellation, we also treat the Johnson–Mehl tessellation, Manhattan grids, nested versions and Palm versions. As part of our proofs, for some planar tessellations, we also derive existence of exponential moments for the number of cells and the number of edges intersecting the unit disk.

## 1 Setting and main results

Random tessellations are a classical subject of stochastic geometry with a very wide range of applications for example in the modeling of telecommunication systems, topological optimization of materials and numerical solutions to PDEs. In this paper we focus on *random planar tessellations* $S \subset \mathbb{R}^2$ which are derived deterministically from a *homogeneous Poisson point process* (PPP) $X = \{X_i\}_{i \in I}$ on $\mathbb{R}^2$ with intensity $0 < \lambda < \infty$. The most famous example here is the *planar Poisson–Voronoi tessellation*.

Since several decades, research has been performed to understand statistical properties of various characteristics of $S$ such as the degree distribution of its nodes, the distribution of the area or the perimeter of its cells, etc. For the classical examples it is usually possible to derive first and second moments for these characteristics as a function of the intensity $\lambda$, see [OBSC09, Table 5.1.1] and for example [M89, M94, MS07]. However, to derive complete and tractable descriptions of the whole distribution of these characteristics is often difficult.

In this paper we contribute to this line of research by proving existence of all exponential moments for the distribution of the total edge length in a unit disc. More precisely, let $B_r \subset \mathbb{R}^2$ denote the closed centered disk with radius $r > 0$ and let $|S \cap A| = \nu_1(S \cap A)$ denote the random total edge length of the tessellation $S \subset \mathbb{R}^2$ in the Lebesgue measurable volume $A \subset \mathbb{R}^2$, where $\nu_1$ denotes the one-dimensional Hausdorff measure. We show for a large class of tessellations that for all $\alpha \in \mathbb{R}$ we have that

$$\mathbb{E}[\exp(\alpha|S \cap B_1|)] < \infty. \tag{1}$$

As a motivation, let us mention that the information on the tail behavior of the distribution of $|S \cap B_1|$ provided by (1) is an important ingredient for example in the large deviations analysis of random tessellations. If additionally the tessellation has sufficiently strong mixing properties, namely that there exists $b > 0$ such that $|S \cap A|$ and $|S \cap B|$ are stochastically independent for measurable sets $A, B \subset \mathbb{R}^2$ with $\mathrm{dist}(A, B) = \inf\{|x - y| \colon x \in A, y \in B\} > b$, then the cumulant generating function

$$\lim_{n \uparrow \infty} n^{-2} \log \mathbb{E}[\exp(-|S \cap B_n|)]$$

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exists, see [HJC18, Lemma 6.1]. This can be used for example to establish the limiting behavior of the percolation probability for the Boolean model with large radii based on Cox point processes where the intensity measure is given by $|S \cap \mathrm{d}x|$, see [HJC18]. Moreover, existence of exponential moments plays a role in giving proper bounds on the acceptable decay of the path-loss function in the context of percolation for the SINR graph based on Cox point processes, see [T18] for details.

Let $\partial A = \bar{A} \setminus A^o$ denote the boundary of a set $A \subset \mathbb{R}^2$ and write $x = (x_1, x_2)$ for $x \in \mathbb{R}^2$. Apart from the classical *Poisson–Voronoi tessellation* (PVT), where

$$S_{\mathrm{V}} = S_{\mathrm{V}}(X) = \bigcup_{i \in I} \partial\{x \in \mathbb{R}^2 \colon |x - X_i| = \inf_{j \in I} |x - X_j|\},$$

and its dual, the *Poisson–Delaunay tessellation* (PDT), where

$$S_{\mathrm{D}} = S_{\mathrm{D}}(X) = \bigcup_{i,j \in I,\, s \in [0,1]} \{sX_i + (1-s)X_j \colon \exists x \in S_{\mathrm{V}}(X) \text{ with } |x - X_i| = |x - X_j| = \inf_{k \in I} |x - X_k|\},$$

we also consider the *Poisson line tessellation* (PLT), where

$$S_{\mathrm{L}} = S_{\mathrm{L}}(X) = \bigcup_{i \in I \colon X_i \in \mathbb{R} \times [0,\pi)} \{x \in \mathbb{R}^2 \colon x_1 \cos X_{i,2} + x_2 \sin X_{i,2} = X_{i,1}\}.$$

As an extension of the PVT also the *Johnson–Mehl tessellation* (JMT) is covered by our results, see for example [BR08]. For this consider the homogeneous PPP $\widetilde{X} = \{(X_i, T_i)\}_{i \in I}$ on $\mathbb{R}^2 \times [0, \infty)$ with intensity $0 < \bar{\lambda} < \infty$ and define the Johnson–Mehl metric by

$$d_{\mathrm{J}}((x,s),(y,t)) = |x - y| + |t - s|, \tag{2}$$

where we use the same notation $|\cdot|$ for the Euclidean norm on $\mathbb{R}^2$ and $[0, \infty)$. Then, the JMT is given by

$$S_{\mathrm{J}} = S_{\mathrm{J}}(\widetilde{X}) = \bigcup_{i \in I} \partial\{x \in \mathbb{R}^2 \colon d_{\mathrm{J}}((x,0),(X_i,T_i)) = \inf_{j \in I} d_{\mathrm{J}}((x,0),(X_j,T_j))\}.$$

Going slightly beyond the setting of Poisson tessellations we also consider the *Manhattan grid* (MG), see for example [HHJC19]. For this let $Y = (Y_{\mathrm{v}}, Y_{\mathrm{h}})$ be the tuple where $Y_{\mathrm{v}} = \{Y_{i,\mathrm{v}}\}_{i \in I_{\mathrm{v}}}$ and $Y_{\mathrm{h}} = \{Y_{i,\mathrm{h}}\}_{i \in I_{\mathrm{h}}}$ are two independent simple stationary point processes on $\mathbb{R}$. We assume throughout this paper that the random variables $\#(Y_{\mathrm{v}} \cap [0,1])$ and $\#(Y_{\mathrm{h}} \cap [0,1])$ have all exponential moments. Then the MG is defined as

$$S_{\mathrm{M}} = S_{\mathrm{M}}(Y) = \bigcup_{i \in I_{\mathrm{v}},\, j \in I_{\mathrm{h}}} (\mathbb{R} \times \{Y_{i,\mathrm{h}}\}) \cup (\{Y_{j,\mathrm{v}}\} \times \mathbb{R}).$$

Note that $S_{\mathrm{M}}$ is stationary, similarly to all previously defined tessellations, however, unlike them, it is not isotropic. One can make $S_{\mathrm{M}}$ isotropic by choosing a uniform random angle in $[0, 2\pi)$, independent of $Y$, and rotating $S_{\mathrm{M}}$ by this angle. Our results for the MG will be easily seen to hold for both the isotropic and anisotropic version of the MG.

Next, let us denote by $(C_i)_{i \in J}$ the collection of cells in the tessellation $S$, where $J = J(S)$. Formally, a cell $C_i$ of $S$ is defined as an open subset of $\mathbb{R}^2$ such that $C_i \cap S = \emptyset$ and $\partial C_i \subset S$. In view of applications, it is sometimes desirable to consider *nested tessellations* (NT), which we can partially also treat with our techniques. For this, let $S_o$ be one of the tessellation processes introduced above, defined via the point process $X^{(o)}$, with cells $(C_i)_{i\in J}$, which now serves as a first-layer process. For every $i \in J$, let $S_i$ be an independent copy of one of the above tessellation processes, maybe of the same type as $S_o$ with potentially different intensity or maybe of a different type, but all $S_i$ should be of the same type. Denote $X^{(i)}$ the underlying independent point process of $S_i$. Then the associated NT is defined as

$$S_{\mathrm{N}} = S_{\mathrm{N}}(X^{(o)}, X^{(1)}, \dots) = S_o \cup \bigcup_{i\in J}(S_i \cap C_i).$$

Here, $\bigcup_{i\in J} S_i \cap C_i$ will be called the second-layer tessellation. This definition of a NT originates from [V09, Section 3.4.4], where this class of tessellations was defined as a special case of *iterated tessellations*.

Finally note that all subgraphs of tessellations having the property (1) inherit this property by monotonicity. In particular, our results cover the cases of the Gabriel graph, the relative neighborhood graph, and the Euclidean minimum spanning tree, since they are subgraphs of the PDT, presented in decreasing order with respect to inclusion.

Having defined the types of tessellations we consider, we can now state our main theorem with its proof and all other proofs presented in Section 2.

**Theorem 1.1.** *We have that* (1) *holds for all* $\alpha \in \mathbb{R}$ *if* $S$ *is a*

*(i) Poisson–Voronoi tessellation,*
*(ii) Johnson–Mehl tessellation,*
*(iii) Poisson–Delaunay tessellation,*
*(iv) Poisson line tessellation, or*
*(v) Manhattan grid.*

Note that using Hölder's inequality and stationarity, the statement of Theorem 1.1 and all subsequent results remain true if $B_1$ is replaced by any bounded measurable subset of $\mathbb{R}^2$.

Let us denote by $(E_i)_{i\in K}$ the collection of edges in the tessellation $S$, where $K = K(S)$. Our proof of Theorem 1.1 for the PVT, JMT and PDT also reveals that exponential moments exist for the number of cells intersecting $B_1$,

$$V = \#\{i \in J\colon C_i \cap B_1 \neq \emptyset\}, \tag{3}$$

and the number of edges intersecting $B_1$,

$$W = \#\{i \in K\colon E_i \cap B_1 \neq \emptyset\}. \tag{4}$$

This is the content of the following corollary.

**Corollary 1.2.** *For the Poisson–Voronoi tessellation and the Johnson–Mehl tessellation, we have for all* $\alpha \in \mathbb{R}$ *that*

$$\mathbb{E}[\exp(\alpha V)] < \infty \tag{5}$$

*and*

$$\mathbb{E}[\exp(\alpha W)] < \infty. \tag{6}$$

*Moreover, for the Poisson–Delaunay tessellation,* (5) *and* (6) *hold for some* $\alpha > 0$.

For the NT, existence of exponential moments for $V$ for the first-layer tessellation can be used to verify (1) for $S_{\mathrm{N}}$. More precisely, we have the following result.

**Corollary 1.3.** *Consider the nested tessellation.*

*(i) If for the first-layer tessellation* (5) *holds for all $\alpha \in \mathbb{R}$ and for the second-layer tessellation* (1) *holds for all $\alpha \in \mathbb{R}$, then also $S_{\mathrm{N}}$ satisfies* (1) *for all $\alpha \in \mathbb{R}$.*

*(ii) If for the first-layer tessellation* (5) *holds for some $\alpha \in \mathbb{R}$ and for the second-layer tessellation* (1) *holds for some $\alpha \in \mathbb{R}$, then also $S_{\mathrm{N}}$ satisfies* (1) *for some $\alpha \in \mathbb{R}$.*

As we will explain in Section 3.5, the statement of Corollary 1.2 is false for the MG based on independent Poisson processes on the axes. However, in the special case, where in the NT is composed of MGs in both layers and the second-layer MG is based on stationary Poisson processes, for this $S_{\mathrm{N}}$ we still obtain (1) for all $\alpha \in \mathbb{R}$. This is the content of the following result.

**Proposition 1.4.** *Consider the nested tessellation and assume that the second-layer tessellation is given by Manhattan grids based on two stationary Poisson processes and the first-layer tessellation is also a Manhattan grid satisfying* (1) *for all $\alpha \in \mathbb{R}$. Then,* (1) *holds for the nested Manhattan grid also for all $\alpha \in \mathbb{R}$.*

Let us mention that for the tessellations studied in Theorem 1.1 considering Palm versions of the underlying point process does not change existence of all exponential moments. We want to be precise here since there are multiple different possibilities to define Palm measures in this context. For the PVT, JMT and PDT, we denote by $X^*$ the Palm version of the underlying unmarked PPP and denote by $S^* = S(X^*)$ its associated tessellations. For the PLT we denote by $X^*$ the Palm version of the underlying PPP only with respect to the first coordinate. This roughly corresponds to $S^*_{\mathrm{L}} = S_{\mathrm{L}}(X^*)$ being distributed as $S_{\mathrm{L}}$ when conditioned to have a line crossing the origin with no fixed angle. The distribution of the Palm version $S^*_{\mathrm{M}} = S_{\mathrm{M}}(Y^*)$ of the MG is given via the distribution of $Y^*$, the Palm version of $(Y_{\mathrm{v}}, Y_{\mathrm{h}})$. Palm distributions of NTs can be defined correspondingly, see for example [HHJC19, V09].

**Corollary 1.5.** *For all tessellations $S$ for which Theorem 1.1 implies* (1) *for all $\alpha \in \mathbb{R}$, we also have for all $\alpha \in \mathbb{R}$ that*

$$\mathbb{E}[\exp(\alpha|S^* \cap B_1|)] < \infty. \tag{7}$$

The remainder of the manuscript is organized as follows. In Section 2 we present the proofs of the statements above. In Section 3 we discuss extensions, relations and limitations of our results and their proofs. We note that in the case when $S$ is a PVT, (1) was verified for small $\alpha > 0$ in [T18] in the context of SINR percolation based on Cox point processes. In view of this line of research, in Section 3.1 and more precisely in Proposition 3.1, we present conditions under which (1) can be guaranteed in the case when the underlying PPP is replaced by some Cox point process.

Further, in Sections 3.2 and 3.4 we discuss relations between the arguments used in the proof of Theorem 1.1 and also connect to prior work. In this context, let us mention that it is a simple consequence of the works [C03, H04] that for all $\alpha \geq 0$

$$\mathbb{E}[\exp(\alpha N^*)] < \infty, \tag{8}$$

where $N^*$ denotes the number of Poisson–Delaunay edges originating from the origin under the Palm distribution for the underlying PPP.

Finally, in Section 3.4 we discuss possible extensions of our results to additively and multiplicatively weighted PVTs and in Section 3.5 we verify the afore mentioned absence of exponential moments for $V$ and $W$ for the MG.

# 2 Proofs

We will use the following notations in this section. We let $Q_r$, respectively $B_r$, denote the box of side length $r > 0$, respectively the closed ball of radius $r > 0$, centered at the origin $o$. For $A \subseteq \mathbb{R}$, we will write $\overline{A}$ for the closure of $A$. For our results, it obviously suffices to consider $\alpha > 0$ instead of $\alpha \in \mathbb{R}$.

## 2.1 Poisson–Voronoi tessellations: Proof of Theorem 1.1 part (i)

We follow the lines of the proof in [T18, Theorem 2.6], where it was shown that $\mathbb{E}[\exp(\alpha|S_{\mathrm{V}} \cap Q_1|)] < \infty$ for some $\alpha > 0$. One main step in this proof was Lemma [T18, Lemma 3.5], which states that, given the distance $R$ between $o$ and the nearest point of $X$ to it, $S_{\mathrm{V}} \cap Q_1$ is determined by $X \cap Q_{c'R}$ for a suitably chosen $c' > 1$. Now we argue that after replacing $Q_1$ with $B_1$, the following stronger version of the aforementioned lemma holds, which allows to handle also the case for all $\alpha$. Note that for the PVT we can identify for each $X_i \in X$ its Voronoi cell $C_i$.

**Lemma 2.1.** *Let $b \geq a > 0$. If $X \cap B_b \neq \emptyset$ we have*

$$|S_{\mathrm{V}} \cap B_a| \leq \sum_{i \in I} \mathbb{1}\{X_i \in B_{b+3a}\} |\partial C_i \cap B_a|. \tag{9}$$

*Proof.* Since $X \cap B_b \neq \emptyset$, we can choose $X_i \in X \cap B_b$. We claim that for any edge of $S_{\mathrm{V}}$ intersecting with $B_a$, the corresponding edge in the dual PDT connects two points in $B_{b+3a}$. Indeed, assume otherwise, then there exists $v \in B_a$, $X_j \in X \cap B_{b+3a}^{\mathrm{c}}$ such that $|v - X_j| = \min\{|v - X_l| \colon l \in I\}$. However, since $v \in B_a$ and $X_i \in B_b$, we have

$$|v - X_i| \leq \max_{y \in B_a, z \in B_b} |y - z| = 2a + (b - a) = b + a,$$

further, since $X_j \in B_{b+3a}^{\mathrm{c}}$,

$$|v - X_j| \geq \mathrm{dist}(\{X_j\}, B_a) > (b + 3a) - a > b + a.$$

This contradicts with the assumption that $|v - X_j| = \min\{|v - X_l| \colon l \in I\}$, hence the claim follows.

Thus, for any Voronoi edge intersecting with $B_a \subseteq B_b$, the corresponding Delaunay edge has both endpoints in $X \cap B_{b+3a}$. In particular, the Voronoi edge is in $\partial C_j$ for some $j$ such that $X_j \in X \cap B_{b+3a}$. The sum in (9) includes the length of the intersection of any such Voronoi edge with $B_a$ among the summands at least once, and this concludes the proof. □

Recall that almost surely, any cell of the PVT is a convex polygon and therefore it is bounded.

**Lemma 2.2.** *For all $a > 0$ and Voronoi cell $C_i$, we have $|\partial C_i \cap B_a| \leq 2\pi a$.*

*Proof.* If $C_i \subseteq B_a$, then $|\partial C_i \cap B_a|$ equals the perimeter of $C_i$, and it is elementary to show that this perimeter is at most $2\pi a$. On the other hand, if $C_i \not\subseteq B_a(x)$, then $C_i \cap B_a$ is a convex polygon since both $C_i$ and $B_a$ are convex. In particular, $|\partial C_i \cap B_a(x)|$ is bounded from above by the perimeter of $C_i \cap B_a$, which is again at most $2\pi a$. □

Using Lemma 2.1 and Lemma 2.2, we have the following immediate consequence, which roughly states that, in case of the PVT, the total edge length in a ball is dominated by a constant times the number of Poisson points in a larger ball, if existence of points is guaranteed.

**Corollary 2.3.** *Let $b \geq a > 0$. Then, if $X \cap B_b \neq \emptyset$, we have that $|S_{\rm V} \cap B_a| \leq 2\pi a \#(X \cap B_{b+3a})$.*

The proof of Theorem 1.1 for the PVT now rests on the idea that it is exponentially unlikely to have large void spaces, using the existence of all exponential moments for Poisson random variables.

*Proof of Theorem 1.1 part (i).* Let

$$R = \inf\{r > 0 \colon B_r \cap X \neq \emptyset\} \tag{10}$$

denote the distance of the closest point in $X$ to the origin. Then we have for all $r > 0$ that

$$\mathbb{P}(R \geq r) = \exp(-\lambda r^2 \pi). \tag{11}$$

Note that by definition, $\mathbb{P}(\#(X \cap B_R) = 1) = 1$. Now, in the event $\{R \leq 1\}$ we have that $B_1 \cap X \neq \emptyset$, and therefore by Corollary 2.3 applied for $a = b = 1$ and $x = o$, we obtain

$$|S_{\rm V} \cap B_1| \leq 2\pi \#(X \cap B_4). \tag{12}$$

On the other hand, in the event $\{R > 1\}$, we can apply Corollary 2.3 for $x = o$, $a = 1$ and $b = R$ to obtain that

$$\begin{aligned} |S_{\rm V} \cap B_1| &\leq 2\pi \#(X \cap B_{R+3}) \leq 2\pi\big(\#(X \cap B_R) + \#(X \cap (B_{R+3} \setminus B_R))\big) \\ &= 2\pi\big(1 + \#(X \cap B_{R+3} \setminus B_R)\big), \end{aligned} \tag{13}$$

almost surely. Therefore, for $\alpha > 0$ we have

$$\exp(\alpha |S_{\rm V} \cap B_1|) \leq \exp(2\pi\alpha \#(X \cap B_4)) \mathbb{1}\{R \leq 1\} + \mathrm{e}^{2\pi\alpha} \exp(2\pi\alpha\#(X \cap (B_{R+3} \setminus B_R)))\mathbb{1}\{R > 1\}, \tag{14}$$

and Theorem 1.1 part (i) follows as soon as we verify that the right-hand side of (14) has all exponential moments.

Note that so far we have not used that $X$ is a Poisson point process, only that $\mathbb{P}(\#(X \cap B_R) = 1) = 1$, which holds for a large class of simple point processes. Now we estimate (14) using particular properties of the PPP $X$. First, since $\#(X \cap B_4)$ is Poisson distributed with parameter $16\pi\lambda$, the expectation of the first term in (14) is finite for all $\alpha > 0$. As for the second term, note that conditional on $R$, $\#(X \cap (B_{R+3} \setminus B_R))$ is Poisson distributed with parameter $\lambda((R+3)^2 - R^2)\pi = \lambda(2R+6)\pi$, and hence, using the Laplace transform for Poisson random variables,

$$\mathbb{E}[\exp(2\pi\alpha\#(X \cap (B_{R+3} \setminus B_R)))\mathbb{1}\{R > 1\}] \leq \mathbb{E}[\exp\big((2R+6)\lambda(\exp(2\pi\alpha) - 1)\big)]. \tag{15}$$

Finally, note that for any non-negative random variable $Z$ we have $\mathbb{E}(Z) \leq \sum_{k=0}^\infty \mathbb{P}(Z \geq k)$ and thus, for $c = 2\lambda(\exp(2\pi\alpha) - 1)$, (11) implies that

$$\mathbb{E}[\exp(cR)] \leq 1 + \sum_{k=1}^\infty \mathbb{P}\big(\exp(cR) \geq k\big) = 1 + \sum_{k=1}^\infty \mathbb{P}\Big(R \geq \frac{\log k}{c}\Big) = 1 + \sum_{k=1}^\infty \exp\Big(-\frac{\lambda\pi(\log k)^2}{c^2}\Big). \tag{16}$$

Since the right-hand side of (16) is finite, also the right-hand side of (15) is finite, which concludes the proof. □

## 2.2 Johnson–Mehl tessellations: Proof of Theorem 1.1 part (ii)

The arguments for the JMT are similar to the ones used in Section 2.1 for the PVT. To start with, we have the following lemma, which is an analogue of Lemma 2.1 in the Johnson–Mehl case. For $(x,s) \in \mathbb{R}^2 \times [0,\infty)$ and $r > 0$ we write $B_r^{\mathrm{J}}(x,s)$ for the closed ball of radius $r$ around $(x,s)$ in the Johnson–Mehl metric, see (2).

**Lemma 2.4.** *Let $b \geq a > 0$. If $\widetilde{X} \cap B_b^{\mathrm{J}} \neq \emptyset$, then $S_{\mathrm{J}} \cap B_a$ is determined by $\widetilde{X} \cap B_{b+3a}^{\mathrm{J}}$. That is, for any $x \in S_{\mathrm{J}} \cap B_a$, if $j \in I$ is such that $d_{\mathrm{J}}((X_j,T_j),(x,0)) = \inf_{k\in I} d_{\mathrm{J}}((X_k,T_k),(x,0))$, then $(X_j,T_j) \in B_{b+3a}^{\mathrm{J}}$.*

*Proof.* Assume that there exists $i \in I$ such that $(X_i,T_i) \in B_b^{\mathrm{J}}$ and that $S_{\mathrm{J}}$ exhibits an edge having a non-empty intersection with $B_a$, and let $x \in B_a$ be a point of such an edge. Then, using the triangle inequality, since

$$d_{\mathrm{J}}((x,0),(X_i,T_i)) \leq d_{\mathrm{J}}((x,0),(o,0)) + d_{\mathrm{J}}((o,0),(X_i,T_i)) = |x| + d_{\mathrm{J}}((o,0),(X_i,T_i)) \leq a+b,$$

and for any $j \in I$ with $(X_j,T_j) \notin B_{b+3a}^{\mathrm{J}}$, we have

$$\begin{aligned} d_{\mathrm{J}}((x,0),(X_j,T_j)) &= T_j + |x - X_j| \geq (T_j + |X_j|) - |x| > b + 3a - a = b + 2a > b + a \\ &\geq d_{\mathrm{J}}((x,0),(X_i,T_i)), \end{aligned}$$

and the result follows. □

*Proof of Theorem 1.1 part (ii).* We start with two preliminary observations. First, let $\mathcal{E}$ denote the set of (closed) edges of $S_{\mathrm{J}}$. By construction of a JMT, any $E \in \mathcal{E}$ has the property that there exist precisely two points $(X_i,T_i),(X_j,T_j)$ (depending on $E$) such that for all $z \in E$

$$d_{\mathrm{J}}((z,0),(X_i,T_i)) = d_{\mathrm{J}}((z,0),(X_j,T_j)) = \inf_{k\in I} d_{\mathrm{J}}((z,0),(X_k,T_k)).$$

In this case, we will write $E = \big((X_i,T_i);(X_j,T_j)\big)$. We claim that for any finite subset $I_0$ of $I$,

$$\#\{\big((X_i,T_i);(X_j,T_j)\big) \colon i,j \in I_0\} \leq 3\#I_0 \tag{17}$$

holds. Indeed, the set on the left-hand side of (17) is in one-to-one correspondency with the set

$$\mathcal{D}(I_0) = \{i,j \in I_0 \colon \exists \big((X_i,T_i);(X_j,T_j)\big) \in \mathcal{E}\}.$$

Since the JMT is a planar graph, so is its dual, and thus $\mathcal{D}(I_0)$ has cardinality at most $3I_0$ thanks to the Euler formula for planar graphs.

Second, we provide an upper bound on the contribution of any single Johnson–Mehl edge in $|S_{\mathrm{J}} \cap B_1|$. All edges $\big((X_i,T_i);(X_j,T_j)\big) \in \mathcal{E}$ are either hyperbolic arcs or straight line segments, see [OBSC09, Property AW2, page 126]. By convexity, we have that

$$|\big((X_i,T_i);(X_j,T_j)\big) \cap B_1| \leq |\partial B_1| = 2\pi. \tag{18}$$

Now, let us define the closest point to the (space-time) origin in the Johnson–Mehl metric

$$R' = \inf\{r > 0 \colon \exists i \in I \text{ with } d_{\mathrm{J}}((o,0),(X_i,T_i)) \leq r\}. \tag{19}$$

Now, in the event $\{R' \leq 1\}$, we have $B_1^{\mathrm{J}} \cap \widetilde{X} \neq \emptyset$, and thus an application of (18) and Lemma 2.4 for $a = b = 1$ gives

$$|S_{\mathrm{J}} \cap B_1| \leq 2\pi\#\{\big((X_i, T_i); (X_j, T_j)\big) \colon (X_i, T_i), (X_j, T_j) \in B_4^{\mathrm{J}}\}.$$

Thanks to (17), the right-hand side is at most $6\pi\#(\widetilde{X} \cap B_4^{\mathrm{J}})$. On the other hand, in the event $\{R' > 1\}$, we can apply Lemma 2.4 for $a = 1$ and $b = R'$, which together with (18) yields

$$|S_{\mathrm{J}} \cap B_1| \leq 2\pi\#\{\big((X_i, T_i); (X_j, T_j)\big) \colon (X_i, T_i), (X_j, T_j) \in B_{R'+3}^{\mathrm{J}}\}. \tag{20}$$

Now, note that

$$\mathbb{P}(\#(B_{R'}^{\mathrm{J}} \cap \widetilde{X}) = 1) = 1. \tag{21}$$

Thus, since in the JMT almost surely each vertex has degree three, we have almost surely

$$\#\{\big((X_i, T_i); (X_j, T_j)\big) \colon (X_i, T_i) \in B_{R'}^{\mathrm{J}}, (X_j, T_j) \in B_{R'+3}^{\mathrm{J}}\} = 3.$$

Further, thanks to (17), we have

$$\#\{\big((X_i, T_i); (X_j, T_j)\big) \colon (X_i, T_i), (X_j, T_j) \in B_{R'+3}^{\mathrm{J}} \setminus B_{R'}^{\mathrm{J}}\} \leq 3\#(\widetilde{X} \cap (B_{R'+3} \setminus B_{R'})).$$

This implies that the right-hand side of (20) is bounded from above by $6\pi\#(\widetilde{X} \cap (B_{R'+3} \setminus B_{R'})) + 6\pi$. We have arrived at the assertion that for $\alpha > 0$

$$\begin{aligned} \exp(\alpha|S_{\mathrm{J}} \cap B_1|) \leq &\exp(6\pi\alpha\#(\widetilde{X} \cap B_4^{\mathrm{J}}))\mathbb{1}\{R' \leq 1\} \\ &+ \exp(6\pi\alpha\#(\widetilde{X} \cap (B_{R'+3}^{\mathrm{J}} \setminus B_{R'}^{\mathrm{J}})) + 6\pi\alpha)\mathbb{1}\{R' > 1\}. \end{aligned} \tag{22}$$

We now use particular properties of the PPP $\widetilde{X}$. Since $(\widetilde{X} \cap B_4^{\mathrm{J}})$ is Poisson distributed with parameter $\mathrm{Leb}(B_4^{\mathrm{J}})$ (here $\mathrm{Leb}$ denotes the Lebesgue measure on $\mathbb{R}^3$), the first term on the right-hand side of (22) is finite for all $\alpha > 0$. As for the second term, since the Johnson–Mehl metric is equivalent to the Euclidean metric in $\mathbb{R}^3$, there exists $M_0 > 0$ such that

$$\limsup_{r\to\infty} \frac{1}{r^2} \log \mathrm{Leb}(B_{r+3}^{\mathrm{J}} \setminus B_r^{\mathrm{J}}) < M_0.$$

Therefore, we have

$$\mathbb{E}[\exp\big(6\pi\alpha\#(\widetilde{X} \cap (B_{R'+3}^{\mathrm{J}} \setminus B_{R'}^{\mathrm{J}}))\big)] \leq \mathbb{E}[\exp\big(M_0(R')^2(\exp(6\pi\alpha) - 1)\big)].$$

Thus, in order to finish the proof of Theorem 1.1 part (ii), it suffices to show that $(R')^2$ has all exponential moments. For this, note that there exists $\varepsilon_0 > 0$ such that

$$\limsup_{r\to\infty} \frac{1}{r^3} \log \mathbb{P}(R' > r) < -\varepsilon_0. \tag{23}$$

Hence, using that $\mathbb{E}(Z) \leq \sum_{k=0}^\infty \mathbb{P}(Z \geq k)$ holds for any non-negative random variable $Z$, we can estimate for $c > 0$

$$\mathbb{E}[\exp(cR^2)] \leq \sum_{k=1}^\infty \mathbb{P}\big(\exp(cR'^2) \geq k\big) = \sum_{k=1}^\infty \mathbb{P}\Big(R' \geq \frac{\sqrt{\log k}}{\sqrt{c}}\Big) \leq \sum_{k=1}^\infty \exp\Big((-\varepsilon_0 + o(1))\frac{(\log k)^{3/2}}{c^{3/2}}\Big).$$

The right-hand side is finite for all $c > 0$. This implies Theorem 1.1 part (ii). □

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## 2.3 Poisson–Delaunay tessellations: Proof of Theorem 1.1 part (iii)

We will distinguish between different types of edges contributing to $|S_{\mathrm{D}} \cap B_1|$. Let $(X, \mathcal{E})$ denote the random Delaunay graph with vertex set $X$ and edge set $\mathcal{E}$ where for all $E \in \mathcal{E}$ we write $E = (X_i, X_j)$ where $X_i, X_j \in X$ are the edge endpoints. We define three different kinds of edges depending on the position of their endpoints. Let us write for $t > 0$

1 $\mathcal{E}_{1,t} = \{E = (X_i, X_j) \in \mathcal{E} \colon X_i \in B_t, X_j \in B_{4t}\}$

2 $\mathcal{E}_{2,t} = \{E = (X_i, X_j) \in \mathcal{E} \colon X_i \in B_t, X_j \in B_{4t}^c\}$

3 $\mathcal{E}_{3,t} = \{E = (X_i, X_j) \in \mathcal{E} \colon X_i, X_j \in B_t^c\}$.

Note that for any $t > 0$, $\mathcal{E} = \mathcal{E}_{1,t} \cup \mathcal{E}_{2,t} \cup \mathcal{E}_{3,t}$. In particular, we can split $|S_{\mathrm{D}} \cap B_1|$, for any $t > 0$, as

$$|S_{\mathrm{D}} \cap B_1| = \sum_{i=1}^{3} \sum_{E \in \mathcal{E}_{i,t}} |E \cap B_1|. \tag{24}$$

Then, by Hölder's inequality,

$$\mathbb{E}[\exp(\alpha |S_{\mathrm{D}} \cap B_1|)] \le \prod_{i=1}^{3} \mathbb{E}\Big[\exp\Big(3\alpha \sum_{E \in \mathcal{E}_{i,t}} |E \cap B_1|\Big)\Big]^{1/3}$$

and we can deal with the different edge sets separately.

Let us partition $\mathbb{R}^2 \setminus \{o\}$ into $16$ disjoint areas $s_i = \{(r\sin\varphi, r\cos\varphi) \colon (i-1)\pi/8 \le \varphi < i\pi/8, r > 0\}$ and define the random radius

$$T = \inf\{r > 6 \colon X \cap (B_r \setminus B_1) \cap s_i \neq \emptyset \text{ and } X \cap (B_{2r} \setminus B_r) \cap s_i \neq \emptyset \text{ for all } 1 \le i \le 16\}, \tag{25}$$

which is the (random) first radius larger than $6$ such that all target areas in a certain regular dart-board-like partition of $B_{2r}$ have at least one point in $B_r \setminus B_1$ and one in $B_{2r} \setminus B_r$. Since $T$ is almost surely finite, Theorem 1.1 part (iii) follows as soon as we verify that for all $i = 1, 2, 3$, we have

$$\mathbb{E}\Big[\exp\Big(\alpha \sum_{E \in \mathcal{E}_{i,T}} |E \cap B_1|\Big)\Big] < \infty$$

for all $\alpha > 0$.

We first show that the total length of edges with both endpoints close to $o$ has exponential moments for small $\alpha$.

**Proposition 2.5.** *For sufficiently small $\alpha > 0$, $\mathbb{E}[\exp(\alpha \sum_{E \in \mathcal{E}_{1,T}} |E \cap B_1|)] < \infty$.*

*Proof.* Note that for any edge $E \in \mathcal{E}$ we have $|E \cap B_1| \le 2$. Further, the subgraph of the Delaunay tessellation $(X, \mathcal{E})$ induced by $X \cap B_{4t}$ is a simple planar graph, Euler's formula for planar graphs implies that

$$\#\{\mathcal{E} \cap (B_{4t} \times B_{4t})\} \le 3\#\{X \cap B_{4t}\}.$$

Thus, since $\mathcal{E}_{1,t} \subset \mathcal{E} \cap (B_{4t} \times B_{4t})$ for all $t > 0$, it suffices to show that $\mathbb{E}[\exp(\alpha \# X_{4T})] < \infty$ holds for sufficiently small $\alpha > 0$, where for $t > 0$ we write $X_t = X \cap B_t$. Note that $X_t$ is Poisson distributed with parameter $16\pi\lambda t^2$.

Using a union bound and the homogeneity of the Poisson point process $X$, we have for $r > 6$

$$\begin{aligned}\mathbb{P}(T > r) &\le 16\big(\mathbb{P}(X \cap (B_r \setminus B_1) \cap s_1 = \emptyset) + \mathbb{P}(X \cap (B_{2r} \setminus B_r) \cap s_1) = \emptyset)\big)\\ &\le 16\big(\exp(-\lambda(r^2-1)\pi/16) + \exp(-3\lambda r^2\pi/16)\big) \le 32\exp(-\lambda r^2\pi/32),\end{aligned}$$

where we used that $(r-1)^2 > r^2/2$ holds for all $r > \sqrt{2}$. Now we compute for $\alpha > 0$

$$\begin{aligned}\mathbb{E}[\exp(6\alpha\#X_{4T})] &= \sum_{k=7}^{\infty} \mathbb{E}[\exp(6\alpha\#X_{4T})\mathbb{1}\{T \in [k-1,k)\}]\\ &\le \sum_{k=7}^{\infty} \mathbb{E}[\exp(6\alpha\#X_{4k})\mathbb{1}\{T \in [k-1,k)\}]\\ &\le \sum_{k=7}^{\infty} \mathbb{E}[\exp(6\alpha\#X_{4k})\mathbb{1}\{T \ge k-1\}]\\ &\le \sum_{k=7}^{\infty} \big(\mathbb{E}[\exp(12\alpha\#X_{4k})]\mathbb{P}(T \ge k-1)\big)^{1/2}\\ &\le 32\sum_{k=7}^{\infty} \exp\big(16k^2\pi\lambda(\exp(12\alpha)-1)\big)^{1/2}\exp\big(-\lambda(k-1)^2\pi/32\big)^{1/2}\\ &= 32\sum_{k=7}^{\infty} \exp\Big(\big(16k^2(\exp(12\alpha)-1)-(k-1)^2/32\big)\frac{\lambda\pi}{2}\Big),\end{aligned}$$

where in the second inequality of the second line we used HÃűlder's inequality. The sum is finite for small $\alpha > 0$, namely, if $16(\exp(12\alpha)-1) < 1/32$, i.e., if $\alpha < 24^{-1}\log(513/512) =: \alpha_{\mathrm{c}}$, which concludes the proof. □

Now, we verify that edges with one endpoint relatively close to $o$ and one very far away from $o$ do not contribute to the edge length in $B_1$.

**Lemma 2.6.** *The random variable $\sum_{E\in\mathcal{E}_{2,T}} |E \cap B_1|$ is equal to zero almost surely.*

In the proof of this lemma, we shall use the equivalent construction of the PDT [M94, Section 1.1], according to which for $X_i, X_j \in X$ and $E = (X_i, X_j)$, the assertion that $E \in \mathcal{E}$ is equivalent to the property that there exists $X_k \in X \setminus \{X_i, X_j\}$ such that the interior of the closed ball circumscribed over the points $X_i, X_j, X_k$ contains no point of $X$. We call this ball a *defining ball* of $E$; almost surely with respect to $X$, all edges of the PDT have precisely two defining balls, with $E$ being their common chord.

*Proof of Lemma 2.6.* Let $k \in \mathbb{N}$. In the event $\{T \in [k-1,k)\}$ any edge $E = (X_i, X_j)$ in $\mathcal{E}_{2,T}$ that also intersects $B_1$ has a minimal length given by $4(k-1)-1$. Thus, any defining ball for $E$ has radius at least $2(k-1)-1$. On the other hand, the diameter of any of the outer segments $(B_{2k} \setminus B_k) \cap s_i$ is bounded from above by $3k/2$. Thus, for $k \ge 6$, the defining ball for $E$ covers at least one of the outer segments. But all outer segments contain at least one vertex not equal to $X_i$ or $X_j$. This contradicts the construction of Delaunay edges, and thus such an edge cannot exist. Since $T > 6$ by construction, this implies the claim. □

Next, we show that edges with both endpoints far away do not contribute to the edge length in $B_1$.

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**Lemma 2.7.** *The random variable $\sum_{E\in\mathcal{E}_{3,T}}|E\cap B_1|$ is equal to zero almost surely.*

*Proof.* Let $k\in\mathbb{N}$. In the event $\{T\in\frac{1}{2}[k-1,k)\}$ any edge $E=(X_i,X_j)$ in $\mathcal{E}_{3,T}$ that also intersects $B_1$ has a minimal length given by $2\sqrt{(k-1)^2/4-1}$, which follows from basic geometry. Now, by the construction of Delaunay edges, there are two balls which define the edge $E$. Now, the straight line $l$ obtained by extending $E$ to infinity in both directions does not contain the origin, and hence $\mathbb{R}^2\setminus l$ is the union of two open half-planes one of which does not contain $o$. We consider the defining ball that has center, say $x$, in this half-plane. Note that $x$ lies on a line perpendicular to the edge $E$.

Assume first that the edge-defining ball centered at $x$ contains the origin. Let $s_x$ denote the infinite segment containing $x$. We claim that in this case, for sufficiently large $k$, the area $B_{k/2}\cap s_x$ is completely contained in the defining ball centered at $x$. Indeed, since $|x|\ge 1$, it suffices to check that the diameter of $B_{k/2}\setminus B_1$ is smaller than the minimal radius of the ball, $\sqrt{(k-1)^2/4-1}$. Using basic geometry, we see that this is the case for all $k>7/(8\cos(\pi/8)-4)\approx 1.7501$. But since $B_{k/2}\cap s_x$ contains a vertex unequal to $X_i$ and $X_j$, this is a contradiction again to the construction of Delaunay edges and thus the ball cannot contain the origin.

Let us then assume that the edge-defining ball centered at $x$ does not contain the origin. Then, the ball must be very large. More precisely, by Pythagoras' theorem, $|x|$ must be at least such that $(k-1)^2/4-1+(|x|-1)^2=|x|^2$ and hence, $|x|\ge(k-1)^2/8$. But in this case, for sufficiently large $k$, the area $(B_k\setminus B_{k/2})\cap s_x$ is completely contained in the defining ball centered at $x$, which is again a contradiction in the event that all $32$ areas contain points. To be more precise, this is true once $k$ is so large that the ball centered at $x$ that has a diameter tangential to $B_1$ with both endpoints situated on $\partial B_{(k-1)/2}$ completely contains $s_x$. Using elementary geometry, this holds whenever $2/(k-1)<\cos(\pi/32)$, in particular for all $k\ge 3$. Since $T>6$ by construction, the lemma follows. □

Next, let us write $X^\lambda$ to indicate the intensity $\lambda$ in the underlying PPP and write $S_{\mathrm{D}}^\lambda=S_{\mathrm{D}}(X^\lambda)$. We have the following scaling relation.

**Lemma 2.8.** *Let $\alpha,\lambda,r>0$. Then we have the following identity in distribution*

$$\alpha|S_{\mathrm{D}}^\lambda\cap B_1|=\alpha\big|S_{\mathrm{D}}^{\lambda/r^2}\cap B_r\big|/r.. \tag{26}$$

*Proof.* Since $X^\lambda$, $X^{\lambda/r^2}$ are homogeneous Poisson point processes with intensities $\lambda$, $\lambda/r^2$, respectively, we have that $X^{\lambda/r^2}\cap B_r$ equals $X^\lambda\cap B_1$ in distribution. Thus, $S^{\lambda/r^2}\cap B_r$ is equal to a rescaled version of $S^\lambda\cap B_1$ in distribution where the length of each edge is multiplied by $r$. This implies the statement (26). □

*Proof of Theorem 1.1 part (iii).* Let us fix the pair $(\alpha,\lambda)$. Using Lemma 2.8, it suffices to show that there exists $a>0$ such that

$$\mathbb{E}\Big[\exp\Big(\alpha\big|S_{\mathrm{D}}^{\lambda/a^2}\cap B_a\big|/a\Big)\Big]<\infty \tag{27}$$

for some $a>0$. Thus, we only have to lift Proposition 2.5 from sufficiently small $\alpha$ to all $\alpha$. Let $r>0$ be sufficiently large such that $\alpha/r<\alpha_{\mathrm{c}}$, where $\alpha_{\mathrm{c}}$ was defined at the end of the proof of Proposition 2.5 representing our bound below which the exponential moments exist.

Note that Proposition 2.5 holds also true if $B_1$ is replaced by $B_a$ and $\alpha$ is replaced by $\alpha/a$, for all $a>0$, where we used that the observation ball $B_a$ enters the proof of the proposition only via

$|E \cap B_a| \le 2a$. Further, observe that the value $\alpha_{\rm c}$ is independent of the intensity parameter of the underlying PPP. These together with Lemmas 2.6 and 2.7 imply that for any $\lambda' > 0$, we have

$$\mathbb{E}\Big[\exp\Big(\alpha\big|S_{\rm D}^{\lambda'} \cap B_r\big|/r\Big)\Big] < \infty.$$

Choosing $\lambda' = \lambda/r^2$ implies (27) with $a = r$ everywhere. This concludes the proof. □

## 2.4 Poisson line tessellations: Proof of Theorem 1.1 part (iv)

We use the notation of Section 1. Since for any line $l_i = \{x \in \mathbb{R}^2 \colon x_1 \cos X_{i,2} + x_2 \sin X_{i,2} = X_{i,1}\}$ of $S_{\rm L}$ we have $|l_i \cap B_1| \le 2$, it suffices to show that the number of lines of $S_{\rm L}$ intersecting with $B_1$ has all exponential moments. Now, a line $l_i$ in $\mathbb{R}^2$ intersects with $B_1$ if and only if its distance parameter $X_{i,1}$ is at most one, independently of its angle parameter $X_{i,2}$. By construction, the number of such lines is Poisson distributed with parameter $2\pi\lambda$, which thus has all exponential moments. □

## 2.5 Manhattan grids: Proof of Theorem 1.1 part (v)

By stationarity, it suffices to verify the statement for $Q_1$ instead of $B_1$. Note that for any edge $E$ in $S_{\rm M}$, either $E \cap Q_1 = \emptyset$ or $|E \cap Q_1| = 1$. Since $Y_{\rm v}$ and $Y_{\rm h}$ are independent, it follows that for all $\alpha > 0$, we have

$$\begin{aligned}\mathbb{E}[\exp(\alpha|S_{\rm M} \cap Q_1|)] &= \mathbb{E}[\exp\big(\alpha(\#(Y_{\rm v} \cap Q_1) + \#(Y_{\rm h} \cap Q_1))\big)] \\ &= \mathbb{E}[\exp\big(\alpha\#(Y_{\rm v} \cap [-1/2,1/2])\big)]\mathbb{E}[\exp\big(\alpha\big(\#Y_{\rm h} \cap [-1/2,1/2]\big)\big)]\end{aligned}$$

Thanks to the assumption that $\#(Y_{\rm h} \cap [-1/2,1/2])$ and $\#(Y_{\rm v} \cap [-1/2,1/2])$ have all exponential moments, the assertion follows. Note that an application of Hölder's inequality would give the same result without the independence assumption on the point processes $Y_{\rm v}, Y_{\rm h}$. □

## 2.6 Number of edges and cells: Proof of Corollary 1.2

We start with verifying (6) using different arguments for each of the three tessellations that we consider (JMT, PVT, PDT). Following this, we can easily derive (5) using (6) (in all cases).

*Proof of* (6) *for the JMT for all* $\alpha > 0$*.* The assertion follows from a closer inspection of the proof of Theorem 1.1 part (ii) in Section 2.2. Indeed, the proof of Lemma 2.1 implies that for $b \ge a > 0$, in the event $\{\widetilde{X} \cap B_b^{\rm J} \neq \emptyset\}$, we have almost surely that

$$W \le \#\{\big((X_i,T_i);(X_j,T_j)\big) \colon (X_i,T_i),(X_j,T_j) \in B_{b+3a}^{\rm J}\}. \tag{28}$$

Recalling the distance $R'$ of the closest point of $\widetilde{X}$ to $(o,0)$ from (19) and applying (28) for $a = b = 1$ in the event $\{R' \le 1\}$ and for $a = 1$ and $b = R'$ in the event $\{R' > 1\}$, and using the planarity of the JMT, for $\alpha > 0$ we obtain

$$\exp(\alpha W) \le \exp\big(3\alpha\#(\widetilde{X} \cap B_4^{\rm J})\big)\mathbb{1}\{R' \le 1\} + \exp\big(3\alpha + 3\alpha\#(\widetilde{X} \cap \overline{(B_{R'+3}^{\rm J} \setminus B_{R'}^{\rm J})})\big)\mathbb{1}\{R' > 1\}.$$

Having this, the proof of (5) for all $\alpha > 0$ follows analogously to how the proof of Theorem 1.1 part (ii) was completed having (22). □

*Proof of* (6) *for the PVT for all* $\alpha > 0$. This proof is a combination of different assertions for the JMT: The one of Theorem 1.1 part (ii) and the one of (6) for the JMT for all $\alpha > 0$. Recall that all edges of $S_{\mathrm{V}}$ are straight line segments. Hence, writing $\mathcal{E}$ for the edge set of $S_{\mathrm{V}}$, we have that for any $E \in \mathcal{E}$, $|E \cap B_1| = \nu_1(E \cap B_1) \le 2$. Combining this with Lemma 2.1 and arguing analogously to the proof of (6) for the JMT, we obtain

$$\exp(\alpha W) \le \exp\big(3\alpha\#(X \cap B_4)\big)\mathbb{1}\{R \le 1\} + \exp\big(3\alpha + 3\alpha\#(X \cap \overline{(B_{R+3} \setminus B_R)})\big)\mathbb{1}\{R > 1\}. \tag{29}$$

Now, since $\#(X \cap B_4)$ is Poisson distributed with parameter $16\pi\lambda$, the expectation of the first term is finite for all $\alpha > 0$. As for the second term, conditional on $R$, $\#(X \cap \overline{(B_{R+3} \setminus B_R)})$ is Poisson distributed with parameter $(2R+6)\pi\lambda$ (cf. Section 2.1). Thus, the second term on the right-hand side of (29) is stochastically dominated by $\exp(3\alpha)\exp\big((\exp(3\alpha) - 1)(2R+6)\pi\lambda\big)$. The finiteness of its expectation for all $\alpha > 0$ follows from the existence of all exponential moments of $R$ (cf. (16)). □

*Proof of* (6) *for the PDT for small* $\alpha > 0$. The assertion follows from a closer inspection of the proof of Theorem 1.1 part (iii) in Section 2.3. Using the edge sets $\mathcal{E}_{1,t}, \mathcal{E}_{2,t}, \mathcal{E}_{3,t}$ introduced in the beginning of Section 2.3 for $t > 0$ and the random variable $T$ defined in (25), HÃűlder's inequality implies that it suffices to show that there exists $\alpha > 0$ such that

$$\mathbb{E}\big[\exp\big(\alpha\#\{E \in \mathcal{E}_{i,T} \colon E \cap B_1 \ne \emptyset\}\big)\big] < \infty$$

holds for all $i \in \{1,2,3\}$. For $i = 2,3$, Lemmas 2.6 and 2.7 imply that the sets $\#\{E \in \mathcal{E}_{i,T} \colon E \cap B_1 \ne \emptyset\}$ are in fact empty almost surely. Now, for $i = 1$, we have that

$$\sum_{E \in \mathcal{E}_{1,T}} |E \cap B_1| \le 2\#\{E \in \mathcal{E}_{i,T} \colon E \cap B_1 \ne \emptyset\} \le 2\#\{\mathcal{E} \cap (B_{4T} \times B_{4T})\}.$$

Now, the proof of Proposition 2.5 shows that the expression on the right-hand side has some (but not all) exponential moments. We conclude (6) for small $\alpha > 0$. □

*Proof of* (5) *for the PVT and JMT for all* $\alpha > 0$ *and for the PDT for small* $\alpha > 0$. Note that any edge of the PVT, PDT or JMT that intersects with $B_1$ is adjacent to precisely two cells intersecting with $B_1$, whereas if $W = 0$, then $V = 1$, and thus we have the trivial bound $V \le 2W + 1$. Thus, the assertion (5) for any given $\alpha/2 > 0$ follows from the assertion (6) for the same $\alpha$. □

## 2.7 Nested tessellations: Proof of Corollary 1.3 and Proposition 1.4

*Proof of Corollary 1.3.* We write $S'$ for a fixed tessellation process that equals $S_i$, $i \in J$, in distribution, and we define $V$ according to (3) for the first-layer tessellation $S_o$, so $J$ is associated to $S_o$. For $\alpha, \beta > 0$, let us write

$$M_\alpha = \mathbb{E}[\exp(\alpha|S' \cap B_1|)], \qquad \text{and} \qquad N_\beta = \mathbb{E}[\exp(\beta V)],$$

where $M_\alpha, N_\beta$ are defined as elements of $[0,\infty]$. Then, we need to show (i) that if $M_\alpha < \infty$ and $N_\beta < \infty$ for all $\alpha, \beta > 0$, then $\mathbb{E}[\exp(\gamma|S_{\mathrm{N}} \cap B_1|)] < \infty$ holds for all $\gamma > 0$, and (ii) if there exists $\alpha, \beta > 0$ such that $M_\alpha < \infty$ and $N_\beta < \infty$, then there exists $\gamma > 0$ such that $\mathbb{E}[\exp(\gamma|S_{\mathrm{N}} \cap B_1|)] < \infty$. First, using Hölder's inequality, we can separate the first from the second layer process,

$$\mathbb{E}[\exp(\alpha|S_{\mathrm{N}} \cap B_1|)] \le \mathbb{E}\Big[\exp\Big(2\alpha \sum_{i \in J \colon C_i \cap B_1 \ne \emptyset} |S_i \cap C_i \cap B_1|\Big)\Big]\mathbb{E}[\exp(2\alpha|S_o \cap B_1|)],$$

where by assumption $\mathbb{E}[\exp(2\alpha|S_o \cap B_1|)] < \infty$. For the other factor, note that we can bound

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(2\alpha \sum_{i\in J\colon C_i\cap B_1\neq\emptyset} |S_i\cap C_i\cap B_1|\Big)\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(2\alpha \sum_{i\in J\colon C_i\cap B_1\neq\emptyset} |S_i\cap C_i\cap B_1|\Big)\Big|S_o\Big]\Big]\\ &\leq \mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(2\alpha \sum_{i\in J\colon C_i\cap B_1\neq\emptyset} |S_i\cap B_1|\Big)\Big|S_o\Big]\Big]\\ &= \mathbb{E}\Big[\prod_{i\in J\colon C_i\cap B_1\neq\emptyset} \mathbb{E}\Big[\exp\Big(2\alpha|S_i\cap B_1|\Big)\Big|S_o\Big]\Big]\\ &= \mathbb{E}\big[M_{2\alpha}^V\big] = \mathbb{E}\big[\exp(V\log M_{2\alpha})\big] = N_{\log M_{2\alpha}},\end{aligned}$$

as an inequality in $[0,\infty]$. From this, (i) follows immediately. As for (ii), let us assume that $M_\alpha < \infty$ holds for some $\alpha > 0$ and $N_\beta < \infty$ holds for some $\beta > 0$. Then, the moment generating function $\mathbb{R} \to [0,\infty]$, $\beta \mapsto N_\beta$ is continuous (in fact, infinitely many times differentiable) in an open neighborhood of $0$, which implies that $\lim_{\beta\to 0} N_\beta = N_0 = 1$. Analogous arguments imply that $\lim_{\alpha\to 0}\log M_\alpha = 0$. Hence, there exists $\alpha > 0$ such that $N_{\log M_\alpha} < \infty$, which implies (ii). □

*Proof of Proposition 1.4.* We verify the statement with $B_1$ replaced by $Q_1$ in (1). According to the assumptions of the proposition, let the first-layer tessellation $S_o$ be a MG satisfying (1) for all $\alpha > 0$, and let us write $Y^o = (Y^o_{\mathrm{v}}, Y^o_{\mathrm{h}})$ for the corresponding pair of point processes on $\mathbb{R}$. We can enumerate the points of $Y^o_{\mathrm{v}} \cap [-1/2, 1/2]$ in increasing order as $Y^o_{\mathrm{v}} \cap [-1/2,1/2] = (P_i)_{i=1}^{N_{\mathrm{v}}}$. Similarly, we can enumerate the points of $Y^o_{\mathrm{h}} \cap [-1/2,1/2]$ in increasing order as $Y^o_{\mathrm{h}} \cap [-1/2,1/2] = (Q_j)_{j=1}^{N_{\mathrm{h}}}$. We further write $P_0 = Q_0 = -1/2$ and $P_{N_{\mathrm{v}}+1} = Q_{N_{\mathrm{h}}+1} = 1/2$. Note that $\sum_{i=1}^{N_{\mathrm{v}}+1}(P_i - P_{i-1}) = \sum_{j=1}^{N_{\mathrm{h}}+1}(Q_j - Q_{j-1}) = 1$.

Now, the collection of cells of $S_o$ is given as $(C_{i,j})_{i=1,\dots,N_{\mathrm{v}}+1, j=1,\dots,N_{\mathrm{h}}+1}$, where $C_{i,j}$ is the open rectangle $(P_{i-1},P_i)\times(Q_{j-1},Q_j)$. We write $S_{i,j}$ for the second-layer tessellation corresponding to $S_{\mathrm{N}}$ in the cell $C_{i,j}$ and $Y^{i,j} = (Y^{i,j}_{\mathrm{v}}, Y^{i,j}_{\mathrm{h}})$ for the associated pair of Poisson processes on $\mathbb{R}$. Here, there exist $\lambda_{\mathrm{v}}, \lambda_{\mathrm{h}} > 0$ such that for all $i \in \{1,\dots,N_{\mathrm{v}}+1\}$ and for all $j \in \{1,\dots,N_{\mathrm{h}}+1\}$, $Y^{i,j}_{\mathrm{v}}$ has intensity $\lambda_{\mathrm{v}}$ and $Y^{i,j}_{\mathrm{h}}$ has intensity $\lambda_{\mathrm{h}}$. Now note that for all $i \in \{1,\dots,N_{\mathrm{v}}+1\}$ and for all $j \in \{1,\dots,N_{\mathrm{h}}+1\}$, all vertical edges of $S_{i,j}$ intersect $C_{i,j}$ in a segment of length $P_i - P_{i-1}$ and all horizontal edges of $S_{i,j}$ intersect $C_{i,j}$ in a segment of length $Q_j - Q_{j-1}$. Thus, we obtain that

$$\begin{aligned}|S_{\mathrm{N}}\cap Q_1| = |S_o\cap Q_1| &+ \sum_{i=1}^{N_{\mathrm{h}}+1}(P_i - P_{i-1})\sum_{j=1}^{N_{\mathrm{v}}+1}\#\big(Y^{i,j}_{\mathrm{v}}\cap(Q_{j-1},Q_j)\big)\\ &+\sum_{j=1}^{N_{\mathrm{v}}+1}(Q_j - Q_{j-1})\sum_{i=1}^{N_{\mathrm{h}}+1}\#\big(Y^{i,j}_{\mathrm{h}}\cap(P_{i-1},P_i)\big).\end{aligned}$$

By HÃűlder's inequality, it suffices to verify the existence of all exponential moments for each of the three terms on the right-hand side separately. The first term has all exponential moments thanks to the assumption of Proposition 1.4. Further, by symmetry between the second and the third term, it suffices to show existence of all exponential moments for one of them; we will consider the second term.

Since, for fixed $i \in \{1,\dots,N_{\mathrm{h}}+1\}$, $\#(Y^{i,j}_v \times (Q_{j-1},Q_j))_{j=1,\dots,N_{\mathrm{v}}+1}$ are independent Poisson random variables with parameters summing up to $\lambda_{\mathrm{v}}$, it follows that their superposition $N_i = \sum_{j=1}^{N_{\mathrm{v}}+1}\#(Y^{i,j}_{\mathrm{v}}\cap(Q_{j-1},Q_j)$ is a Poisson random variable with parameter $\lambda_{\mathrm{v}}$. Further, conditional on $(P_i)_{i=1}^{N_{\mathrm{h}}}$, $(N_i)_{i=1}^{N_{\mathrm{h}}+1}$ are independent.

Now, fix $\alpha > 0$, and let $K_\alpha > 0$ be such that for all $x \in (-\infty, \alpha]$ we have $\exp(x) - 1 \le K_\alpha x$. Using that $P_i - P_{i-1} \le 1$ for all $i$ and $\sum_{i=1}^{N_{\rm h}+1}(P_i - P_{i-1}) = 1$, we estimate

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\alpha \sum_{i=1}^{N_{\rm h}+1}(P_i - P_{i-1}) \sum_{j=1}^{N_{\rm v}+1} \#(Y_{\rm v}^{i,j} \cap (Q_{j-1}, Q_j))\Big)\Big] = \mathbb{E}\Big[\exp\Big(\alpha \sum_{i=1}^{N_{\rm h}+1}(P_i - P_{i-1})N_i\Big)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(\alpha \sum_{i=1}^{N_{\rm h}+1}(P_i - P_{i-1})N_i\Big)\Big|(P_i)_{i=1}^{N_{\rm h}}\Big]\Big] = \mathbb{E}\Big[\prod_{i=1}^{N_{\rm h}+1}\mathbb{E}\Big[\exp\Big(\alpha(P_i - P_{i-1})N_i\Big)\Big|(P_i)_{i=1}^{N_{\rm h}}\Big]\Big] \\
&= \mathbb{E}\Big[\prod_{i=1}^{N_{\rm h}+1}\exp\big(\lambda_{\rm v}\exp(\alpha(P_i - P_{i-1}) - 1)\big)\Big] \le \mathbb{E}\Big[\prod_{i=1}^{N_{\rm h}+1}\exp\big(K_\alpha\lambda_{\rm v}\alpha(P_i - P_{i-1})\big)\Big] \\
&= \mathbb{E}\Big[\exp\Big(\sum_{i=1}^{N_{\rm h}+1}K_\alpha\lambda_{\rm v}\alpha(P_i - P_{i-1})\Big)\Big] = \exp\big(K_\alpha\lambda_{\rm v}\alpha\big).
\end{aligned}$$

Since the right-hand side is finite (note that it is even non-random), we conclude the proposition. □

## 2.8 Palm versions of tessellations: Proof of Corollary 1.5

We handle each case separately.

*Proof of Corollary 1.5 for the PVT.* Corollary 1.5 follows directly from Lemma 2.1 and the Slivnyak–Mecke theorem. Indeed, since Lemmas 2.1 and 2.2 use no information about the distribution of $X$ but only the definition of a Voronoi tessellation, these lemmas remain true after replacing $S^*$ by $S$. Next, the Palm version $X^*$ of the underlying PPP equals $X \cup \{o\}$ in distribution by the Slivnyak–Mecke theorem, in particular, it contains $o$ almost surely. Thus, using the aforementioned versions of Lemmas 2.1 (for $a = b = 1$) and 2.2, we deduce that $|S \cap B_1|$ is stochastically dominated by $2\pi(\#(X \cap B_4) + 1)$. This random variable has all exponential moments, hence the corollary. □

*Proof of Corollary 1.5 for the JMT.* This is analog to the proof for the PVT where instead of the Lemmas 2.1 and 2.2 we use the Lemma 2.4 and equation (18). □

*Proof of Corollary 1.5 for the PDT.* We verify Corollary 1.5 via a straightforward geometric argument given Theorem 1.1 part (iii) and the assertion (8) originating from [C03, H04]. (We expect that an alternative proof using arguments of the proof of the theorem is possible, see Section 3.2.) Indeed, recalling $N^*$ defined before (8) and the fact that $S^*$ equals the Delaunay tessellation of $X \cup \{o\}$ in distribution, we verify the following lemma. Its proof uses the characterization of Delaunay edges explained after Lemma 2.6 in Section 2.3.

**Lemma 2.9.** *Almost surely with respect to $X$, all edges in the Delaunay tessellation of $X \cup \{o\}$ that are not contained in the Delaunay tessellation of $X$ connect $o$ to one of its Delaunay neighbors in $X \cup \{o\}$.*

*Proof.* The following statements hold for almost all realizations of $X$. Let us assume that $E$ is an edge in the Delaunay tessellation of $X \cup \{o\}$. If $E = (o, X_i)$ for some $X_i \in X$, then there is nothing to verify. Else, $E = (X_i, X_j)$ for some $X_i, X_j \in X$. Now, the Delaunay edge $E$ is the common chord of precisely two Delaunay triangles in $X \cup \{o\}$, the third vertex of at least one of them not being equal to $o$. But this means that the open out-circle of this triangle contains no point of $X \cup \{o\}$, in particular no point of $X$, and thus this triangle also exists in the Delaunay tessellation of $X$. Hence, $E$ is contained in the Delaunay tessellation of $X$. □

Thanks to Lemma 2.9 and the fact that for any edge $E$ of $S^*$ we have $|E \cap B_1| \leq 2$, we have $|S^* \cap B_1| \leq |S \cap B_1| + 2N^*$. Now, $|S \cap B_1|$ has all exponential moments thanks to Theorem 1.1 part (iii) and $N^*$ has all exponential moments thanks to the main results of [C03, H04], hence the corollary. □

*Proof of Corollary 1.5 for the PLT.* According to the definition of the Palm version $S^*$ that we provided in Section 1, $S^*$ equals $S_L(X^{**})$ where $X^{**} = X \cup \{(0, \Phi)\}$, with $\Phi$ being a uniform random angle in $[0, \pi)$ that is independent of $X$. (Note that here we also used the Slivnyak–Mecke theorem.) Thus, $S^* = S \cap \{l\}$, where $l = \{x \in \mathbb{R}^2 \colon x_1 \cos \Phi + x_2 \sin \Phi = 0\}$. Since the intersection of $l$ with $B_1$ has length 2, the corollary in the case of a PLT follows directly from Theorem 1.1 part (iv). □

*Proof of Corollary 1.5 for the MG.* We verify the statement with $B_1$ replaced by $Q_1$, which is sufficient thanks to the stationarity of $Y_{\mathrm{v}}$ and $Y_{\mathrm{h}}$. First, let us write $Y_{\mathrm{v}}^*$ and $Y_{\mathrm{h}}^*$ for the Palm versions of $Y_{\mathrm{v}}$ and $Y_{\mathrm{h}}$. Here, $Y_{\mathrm{v}}^*$ is defined via the property [HJC18, Section 2.2] that

$$\mathbb{E}\big[f(Y_{\mathrm{v}}^*)\big] = \mathbb{E}\Big[\frac{1}{\lambda_{\mathrm{v}}} \sum_{X_i \in Y_{\mathrm{v}} \cap [-1/2, 1/2]} f(Y_{\mathrm{v}} - X_i)\Big]$$

for any measurable $f$ taking the set of $\sigma$-finite counting measures on $\mathbb{R}$ to $[0, \infty)$. Then the Palm version $Y^*$ is given as [HHJC19, Section III.B]

$$S_{\mathrm{M}}^* = (Y_{\mathrm{v}} \times \mathbb{R}, Y_{\mathrm{h}}^* \times \mathbb{R})\mathbb{1}\Big\{U \leq \frac{\lambda_{\mathrm{h}}}{\lambda_{\mathrm{h}} + \lambda_{\mathrm{v}}}\Big\} + (Y_{\mathrm{v}}^* \times \mathbb{R}, Y_{\mathrm{h}} \times \mathbb{R})\mathbb{1}\Big\{U > \frac{\lambda_{\mathrm{h}}}{\lambda_{\mathrm{h}} + \lambda_{\mathrm{v}}}\Big\}$$

where $U$ is a uniformly distributed random variable on $[0, 1]$ that is independent of $S_{\mathrm{M}}$. Now, we verify that $Y_{\mathrm{v}}^* \times [-1/2, 1/2]$ and $Y_{\mathrm{h}}^* \times [-1/2, 1/2]$ have all exponential moments. Using these and the mutual independence of $Y_{\mathrm{v}}$, $Y_{\mathrm{h}}$, and $U$, the proof of Corollary 1.5 for the MG can be completed analogously to the proof of Theorem 1.1 part (v) in Section 2.5. We only consider $Y_{\mathrm{v}}^*$, the proof for $Y_{\mathrm{h}}^*$ is analogous. For $\alpha > 0$ we have

$$\begin{aligned}
&\mathbb{E}[\exp\big(\alpha \#(Y_{\mathrm{v}}^* \cap [-1/2, 1/2])\big)] = \mathbb{E}\Big[\frac{1}{\lambda_{\mathrm{v}}} \sum_{X_i \in Y_{\mathrm{v}} \cap [0,1]} \exp\big(\alpha \#((Y_{\mathrm{v}} - X_i) \cap [-1/2, 1/2])\big)\Big] \\
&= \mathbb{E}\Big[\frac{1}{\lambda_{\mathrm{v}}} \sum_{X_i \in Y_{\mathrm{v}} \cap [0,1]} \exp\big(\alpha \#(Y_{\mathrm{v}} \cap [X_i - 1/2, X_i + 1/2])\big)\Big] \\
&\leq \mathbb{E}\Big[\frac{1}{\lambda_{\mathrm{v}}} \sum_{X_i \in Y_{\mathrm{v}} \cap [0,1]} \exp\big(\alpha \#(Y_{\mathrm{v}} \cap [-1, 1])\big)\Big] = \frac{1}{\lambda_{\mathrm{v}}} \mathbb{E}\big[\#(Y_{\mathrm{v}} \cap [0, 1]) \exp\big(\alpha \#(Y_{\mathrm{v}} \cap [-1, 1])\big)\big] \\
&\leq \frac{1}{\lambda_{\mathrm{v}}} \mathbb{E}[\#(Y_{\mathrm{v}} \cap [0, 1])^2]^{1/2} \mathbb{E}[\exp\big(2\alpha \#(Y_{\mathrm{v}} \cap [-1, 1])\big)]^{1/2} < \infty,
\end{aligned}$$

where in the first inequality of the last line we used HÃűlder's inequality. □

## 3 Discussion

In this section we discuss extensions, relations and limitations of our statements in Section 1 and their corresponding proofs in Section 2.

### 3.1 Extensions of Theorem 1.1 to Cox–Voronoi tessellations

So far we have limited our attention to random tessellations defined via a deterministic rule applied to a random collection of points given by a PPP, i.e., $S = S(X)$ where $X$ is a stationary PPP. It is natural to ask under what conditions existence of exponential moments for the total edge length in the unit disk can be guaranteed for tessellations $S(\mathcal{X})$ where $\mathcal{X}$ is not a PPP but some different stationary planar point process. As a starting point for future studies, in this section we give an answer to this question for the Voronoi tessellation based on a stationary Cox point process (CPP) $\mathcal{X}$. Here, a *Cox point process* is a PPP with random intensity measure $\Lambda(\mathrm{d}x)$, see for example [DVJ08] for details. We have the following proposition.

**Proposition 3.1.** *Consider $S_{\mathrm{V}}(\mathcal{X})$ where $\mathcal{X}$ is a stationary Cox point process with intensity measure $\Lambda$ satisfying*

$$\limsup_{|B|\uparrow\infty} \frac{1}{|B|} \log \mathbb{E}\big[\exp\big(\alpha\Lambda(B)\big)\big] = f(\alpha) \tag{30}$$

*for all $\alpha \in \mathbb{R}$ for some function $f$ with $f(\alpha) < 0$ if $\alpha < 0$ and $f(\alpha) < \infty$ if $\alpha > 0$. Then, for $S = S_{\mathrm{V}}(\mathcal{X})$,* (1) *holds for all $\alpha \in \mathbb{R}$.*

*Proof of Proposition 3.1.* By translation invariance we can give a proof for $Q_1$ replacing $B_1$. First note that by construction, the Voronoi cells of any simple point process are convex. Hence, Lemmas 2.1, 2.2 and subsequently Corollary 2.3 in Section 2.1, also hold for $S_{\mathrm{V}}(\mathcal{X})$ based on the CPP $\mathcal{X}$. Now, to accommodate the Cox process, we give a slightly different proof based on the same ideas as used in the proof of Theorem 1.1 part (i) in Section 2.1. Let

$$K = \inf\{k \in \mathbb{N}_0 \colon \mathcal{X} \cap Q_{k+1} \neq \emptyset\}.$$

Then for $\alpha > 0$, we have that

$$\begin{aligned}
\mathbb{E}[\exp(\alpha|S_{\mathrm{V}}(\mathcal{X}) \cap Q_1|)] &= \sum_{k\ge 0} \mathbb{E}[\exp(\alpha|S_{\mathrm{V}}(\mathcal{X}) \cap Q_1|)\mathbb{1}\{K = k\}] \\
&\le \sum_{k\ge 0} \mathbb{E}[\exp(4\alpha\#(\mathcal{X} \cap Q_{k+5} \setminus Q_k))\mathbb{1}\{K = k\}] \\
&\le \sum_{k\ge 0} \mathbb{E}\big[\exp\big((\exp(4\alpha) - 1)\Lambda(Q_{k+5} \setminus Q_k) - \Lambda(Q_k)\big)\big] \\
&\le \sum_{k\ge 0} \mathbb{E}\big[\exp\big(2(\exp(4\alpha) - 1)\Lambda(Q_{k+5} \setminus Q_k)\big)\big]^{1/2} \mathbb{E}\big[\exp\big(-2\Lambda(Q_k)\big)\big]^{1/2}
\end{aligned}$$

where we used Corollary 2.3 for the second line, the Laplace transform for PPPs for the third line and Hölder's inequality for the fourth line. Now, under the assumption (30), for sufficiently large $k$,

$$\mathbb{E}\big[\exp\big(-2\Lambda(Q_k)\big)\big] \le \exp(c_1 k^2 f(-2))$$

for some $c_1 \in (0,\infty)$ and for some $c_2 \in (0,\infty)$ also

$$\mathbb{E}\big[\exp\big(2(\exp(4\alpha) - 1)\Lambda(Q_{k+5} \setminus Q_k)\big)\big] \le \exp\Big(c_2(10k + 25) f\big(2(\exp(4\alpha) - 1)\big)\Big)$$

and thus $\mathbb{E}[\exp(\alpha|S_{\mathrm{V}}(\mathcal{X}) \cap Q_1|)] < \infty$. $\square$

The condition (30) holds for all $\alpha \in \mathbb{R}$ for example for almost-surely bounded random intensity measures where for some $c > 0$ almost surely $\Lambda(B) \le c|B|$. A relevant example here is the *modulated Poisson point process* where $\Lambda(\mathrm{d}x) = (\lambda_1 \mathbb{1}\{x \in \Xi\} + \lambda_2 \mathbb{1}\{x \in \Xi^{\mathrm{c}}\})\mathrm{d}x$, with $\Xi$ being a stationary random closed set, e.g., a Poisson–Boolean-model, and $\lambda_1, \lambda_2 \ge 0$, see [CSKM13, Section 5.2.2].

Another example for which condition (30) holds, and which is unbounded, is the shot-noise field, see [CSKM13, Section 5.6], where $\Lambda(\mathrm{d}x) = \mathrm{d}x \sum_{i\in I} \kappa(x - Y_i)$ for some integrable kernel $\kappa\colon \mathbb{R}^2 \to [0,\infty)$ with compact support and $\{Y_i\}_{i\in I}$ a stationary PPP. Indeed, for the shot-noise field, $\Lambda(B) = \sum_{i\in I} \int \mathrm{d}x\kappa(x)\mathbb{1}\{x \in B(Y_i)\} \le \#(Y_i \in C) \int \mathrm{d}x\kappa(x)$ with $C = C' \oplus B$ where $C'$ denotes the support of $\kappa$ . Now, $Z = \#(Y_i \in C)$ is a Poisson random variable and we denote its parameter $\rho$. Then,

$$\begin{aligned}\limsup_{|B|\uparrow\infty} |B|^{-1} \log \mathbb{E}[\exp(\alpha\Lambda(B))] &\le \rho(\exp(\alpha \int \mathrm{d}x\kappa(x)) - 1) \limsup_{|B|\uparrow\infty} \frac{|C|}{|B|} \\ &= \rho(\exp(\alpha \int \mathrm{d}x\kappa(x)) - 1)\end{aligned}$$

has the desired property.

Note that, more generally, the condition (30) holds if $\Lambda$ is $b$-dependent and $\Lambda(Q_1)$ has all exponential moments. Here, for $b > 0$, we call $\Lambda$ $b$-dependent if for any two measurable sets $A, B \subset \mathbb{R}^2$ such that $\mathrm{dist}(A, B) = \inf_{x\in A, y\in B} |x - y| > b$, the restrictions $\Lambda|_A$ and $\Lambda|_B$ of $\Lambda$ to $A$ respectively $B$ are independent. Indeed, by stationarity of $\Lambda$, it suffices to verify (30) for $B = Q_k$ in the limit $\mathbb{N} \ni k \to \infty$. Let us assume that $\Lambda$ is $b$-dependent. Then, for fixed $k$, we can partition $Q_k$ into a bounded number of disjoint subsets such that each of these subsets consists of (apart from the boundaries) disjoint copies of $Q_1$ and the restrictions of $\Lambda$ to these copies are pairwise independent. Using this independence and the existence of all exponential moments of $\Lambda(Q_1)$, further applying HÃűlder's inequality for the collection of partition sets, (30) follows. Note that the shot-noise field is $b$-dependent, and so is the modulated Poisson process if $\Xi$ is a Poisson–Boolean model.

## 3.2 Relation of Theorem 1.1 part (iii) to prior work

Consider the PDT $S_{\mathrm{D}}$. Using the notation of Section 1, let us consider the assertions

$$\forall\alpha > 0\colon \mathbb{E}[\exp(\alpha N^*)] < \infty, \tag{31}$$

$$\forall\alpha > 0\colon \mathbb{E}[\exp(\alpha|S^*_{\mathrm{D}} \cap B_1|)] < \infty \text{ and} \tag{32}$$

$$\forall\alpha > 0\colon \mathbb{E}[\exp(\alpha|S_{\mathrm{D}} \cap B_1|)] < \infty. \tag{33}$$

Recall that (31) is a simple consequence of the main results of [C03, H04]. Further, given (31) and (33) and the coupling $X^* = X \cup \{o\}$, a simple geometric argument implies (32), see Section 2.8. A similar geometric proof could be provided in order to deduce (33) from (32) using (31). However, we did not find a way to derive (32) from (31). We encountered two main difficulties: First, (31) does not provide sufficient information about $|S \cap B_1|$ if the cell of $o$ in $S^*$ has a very small diameter. Second, even if this cell is so large that it covers $B_1$, edges of $S^*$ connecting two points of $X$ may still cover this cell and in particular also $B_1$.

We chose the alternative route of verifying (33) without using (31), and then putting (33) and (31) together in order to derive (32). We expect that (32) can also be proven using an extended version of our proof for (33), with no reference to (31). However, we chose the proof involving $N^*$ because we found it more straightforward and intuitive.

### 3.3 Relation of the proof of Theorem 1.1 part (iii) to part (i)

Lemma 2.8 is easily seen to hold also for the PVT instead of the PDT. This gives rise to an alternative way of proving Theorem 1.1 part (i), given the original proof of the same statement for small $\alpha$ in [T18, Section 3.1.2]. Indeed, it is easy to see that the original proof for small $\alpha$ works also with $Q_1$ replaced by $Q_a$ and $\alpha$ replaced by $\alpha/a$ for $a > 0$. Therefore, given this proof, the proof of Theorem 1.1 part (iii)can be completed similarly to how we completed the proof of Theorem 1.1 part (iii) given Proposition 2.5 and Lemmas 2.6, 2.7, 2.8.

In contrast, the alternative proof that we provided for Theorem 1.1 part (i) in Section 2.1 is an improved version of the proof for small $\alpha$ in [T18]. Section 2.1 shows that if we use the $\ell^2$-norm, which is the natural norm for the PVT, it is possible to control the influence of far away Poisson points better. More precisely, the proof clarifies that given the distance $R$ of the nearest point to the origin, all Poisson points that are decisive for the Voronoi tessellation in $B_1$ come from the annulus $B_{R+3} \setminus B_R$ (given that $R$ is sufficiently large). We note that such a statement is not true for the PDT. Indeed, assume that $B_1$ contains exactly one point $X_i$ of $X$. Then $X_i$ has degree at least 3 in the Delaunay graph, all edges connecting $X_i$ to one of its neighbors intersect with $B_1$, and given the norm of the closest neighbor, the norm of the second-closest neighbor can be arbitrarily large.

### 3.4 Extensions to additively and multiplicatively weighted PVTs

Let us mention that the construction of the PVT can be substantially generalized, giving rise to the *additively and multiplicatively weighted PVT* (aPVT, mPVT), also see [OBSC09]. For this, as in the construction of the JMT, consider the homogeneous PPP $\widetilde{X} = \{(X_i, T_i)\}_{i\in I}$ on $\mathbb{R}^2 \times [0,\infty)$ and define distance mappings from $\mathbb{R}^2 \times (\mathbb{R}^2 \times [0,\infty))$ to $[0,\infty)$ by

$$d_{\mathrm{a}}^{\varphi}(x,(y,t)) = \varphi(|x-y|) + t, \qquad \text{and} \qquad d_{\mathrm{m}}^{\varphi}(x,(y,t)) = t^{-1}\varphi(|x-y|)$$

for some strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$. The associated tessellations are given by

$$S_{\mathrm{aV}} = S_{\mathrm{aV}}(\widetilde{X}) = \bigcup_{i\in I} \partial\{x \in \mathbb{R}^2 \colon d_{\mathrm{a}}^{\varphi}(x,(X_i,T_i)) = \inf_{j\in I} d_{\mathrm{a}}^{\varphi}(x,(X_j,T_j))\} \text{ and}$$
$$S_{\mathrm{mV}} = S_{\mathrm{mV}}(\widetilde{X}) = \bigcup_{i\in I} \partial\{x \in \mathbb{R}^2 \colon d_{\mathrm{m}}^{\varphi}(x,(X_i,T_i)) = \inf_{j\in I} d_{\mathrm{m}}^{\varphi}(x,(X_j,T_j))\}.$$

For the additively weighted case with $\varphi(x) = x$, we for example recover the JMT. For $\varphi(x) = x^2$, the associated tessellation is referred to as the Laguerre tessellation, see [LZ08]. Note that the aPVT and mPVT can exhibit substantially different behavior than the PVT. For examples, cells can be empty, cells may not contain their nucleus, or cells may be disconnected. Our proof technics for the JMT can be used to cover those cases where the distance mapping is a metric, given that the probability that $\widetilde{X}$ contains no point of a ball of radius $r$ in the corresponding metric decays at least as $\exp(-cr^{-(2+\varepsilon)})$ in the limit $r \to \infty$, for some $\varepsilon > 0$. This is indeed the case for example for the JMT (where one can choose $\varepsilon = 1$, cf. (23)), but for instance not for the Laguerre tessellation.

### 3.5 Absence of exponential moments for the number of edges and cells

In Corollary 1.2, we provide statements about existence of exponential moments for $V$, the number of cells intersecting $B_1$, and $W$, the number of edges intersecting $B_1$. In this section we want to exhibit one example in our family of tessellations for which exponential moments for $V$ do not exist. Indeed, take the MG where the underlying stationary point processes are PPPs $Y_\mathrm{v}$ and $Y_\mathrm{h}$ with intensity $\lambda$. By translation invariance, we can also consider the random variable $V'$, the number of cells intersecting $Q_1$. Then we have that

$$\begin{aligned}\mathbb{E}[\exp(\alpha V')] &= \mathbb{E}\big[\exp\big(\alpha\big((\#(Y_\mathrm{v}\cap[-1/2,1/2])+1)((\#(Y_\mathrm{h}\cap[-1/2,1/2])+1)\big)\big)\big]\\ &= \exp\big(\alpha-\lambda+2\lambda(\exp(\alpha)-1)\big)\sum_{k\geq 0}\exp\big(\lambda(\exp(\alpha k)-1)\big)\lambda^k/k! = \infty.\end{aligned}$$

Since, for the MG based on PPPs, $V$ and $W$ are of the same order, it follows that $\mathbb{E}[\exp(\alpha W)] = \infty$.

Finally, let us mention that for the PLT, it would be rather interesting to try to derive closed form expressions for the distribution of $V$ using similar techniques as in [C03] for $N^*$, the number of Delaunay edges of a typical Poisson point. Then, these representations could potentially be used to derive asymptotics for the probability of many cells intersecting $B_1$, which could subsequently lead to a proof for the existence of exponential moments for $V$.

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